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Simple nonlinear systems and navigating catastrophes

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Abstract. Tipping points are a common occurrence in complex adaptive systems. In such systems feedback dynamics strongly influence equilibrium points and they are one of the principal concerns of research in this area. Tipping points occur as small changes in system parameters result in disproportionately large changes in the global properties of the system. In order to show how tipping points might be managed we use the Maximum Entropy (MaxEnt) method developed by Jaynes to find the fixed points of an economic system in two different ways. In the first, economic agents optimise their choices based solely on their personal benefits. In the second they optimise the total benefits of the system, taking into account the effects of all agent's actions. The effect is to move the game from a recently introduced dual *localised* Lagrangian problem to that of a single *global* Lagrangian. This leads to two distinctly different but related solutions where localised optimisation provides more flexibility than global optimisation. This added flexibility allows an economic system to be managed by adjusting the relationship between macro parameters, in this sense such manipulations provide for the possibility of "steering" an economy around potential disasters.

1 Introduction

One of the key issues in studying complex systems is understanding those dynamics that might appear to be stable but are subject to rapid, unforeseen and significant shifts in their macroscopic properties [1]. Complex adaptive systems often have many elements that are coupled together through feedback loops and they tend to exhibit unpredictable dynamics as well as showing a consistent long term trend. For example a share market might have unpredictable daily jumps in the individual prices of shares but the overall market might grow at a stead $7\% \pm \sigma$ for years, where σ represents small random fluctuations in price. However share markets can crash without warning, wiping out many years of steady growth. Combining some of these properties in a unified model has proved a difficult, but not yet impossible, task [2]. A very early model of random fluctuations in share markets was introduced by Bachelier in his Ph.D. thesis in 1900 [3,4], pre-dating Einstein's work on what is now called Brownian motion by a number of years.

It is not simply that these systems are complicated with many interacting elements, although this is often a contributing factor. For example it has been shown that even the simplest systems with only two interacting elements can show chaotic dynamics [5]. However in many of these systems we can usefully distinguish between the fast dynamics that are difficult to predict and the slower long term but more predictable dynamics, and it is these slower dynamics, specifically their fixed points and their tipping points, that are the topic of this work.

The aim of this paper is to provide two alternative interpretations of the same economic scenario, both of which can be analysed in terms of the MaxEnt methodology introduced by Jaynes [6]. In the first scenario a recently developed MaxEnt Lagrangian formulation of *micro-economic* game theory is described [7]. The equilibrium solutions of such a formulation has previously been shown to have analogues in drift-diffusion models of behavioural game theory [8,9]. An alternative Lagrangian is then introduced where each agent is trying to optimise a global utility function, extending the previous analysis to a *macro-economic* Lagrangian. This interpretation has analogous results in previous work on drift diffusion processes applied to systems where both players have the same utility function [10]. We provide three examples of different economic scenarios that give rise to striking differences in the nature of their tipping points and the techniques that can be employed to manage them.

One possible solution for addressing tipping points is to reverse the direction of the "parametric drift" that is driving the system to collapse before the tipping point is crossed, and thereby return the system to its original state. However reversing such a dynamic may be too difficult or even if the effort is successful the overall system remains susceptible to the same tipping point in the future. In this context it is not necessarily the final position the system finds itself in that is the problem, it is that the abrupt shift in the system is too quick for suitable adjustments to be

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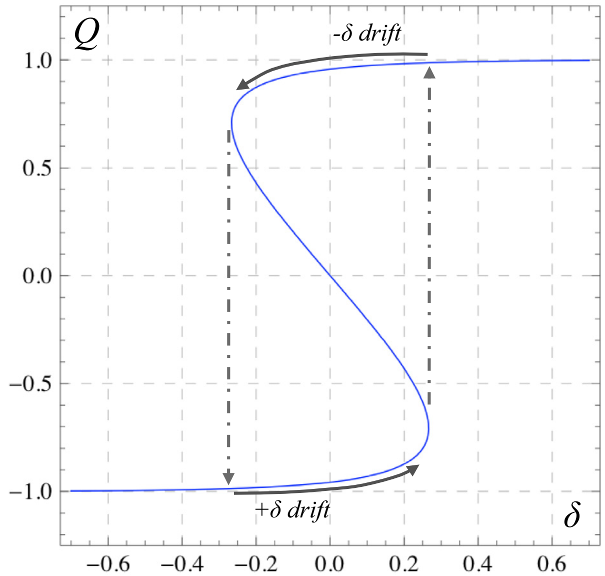

Fig. 1. Hysteresis and the cyclical collapse of a system that is drifting across an equilibrium surface. Q can be thought of as a behavioural outcome, dictated by the system structure and its parameters.

made: for example rising sea levels may be something we can adapt to if it happens over many thousands of years, but over decades the effects could be catastrophic for low lying coastal populations. Here we suggest that by managing the macro-parameters of dynamical systems such collapses might be avoided. A key result is that by breaking up a system's optimisation so that it reflects locally optimised solutions, the resultant control of parameters allows for a more finessed control of the overall system dynamics and the ability to steer around regions where catastrophic collapses might occur.

2 A toy model of tipping points and stochastic dynamics

A tipping point is where a system parameter that can be thought of as fixed over the short term slowly varies over a longer time frame and as it varies the system passes through a bifurcation where there is no longer a locally feasible equilibrium of the system and it collapses to the next nearest fixed point, one that can be very far away in phase space. Figure 1 illustrates this situation by plotting the equilibrium points of the equation $Q = \tanh(2(Q+\delta))$, $Q \in [-1, 1]$ where equilibrium points are any Q satisfying this equation. In this plot it is clear that there are either one or three equilibrium values of Q for any given value of δ. Such an equation comes from the analysis of certain dynamical systems where many individual elements are stochastically interacting with one another, such as a lattice of particles [11] or models of people's social interactions [12].

Such systems are stochastic in nature and so Q is a random variable and the plot of Q in Figure 1 is related to the central tendency of Q, see references [13,14]. In this sense the stochastic variation of the system around a central value of Q can be thought of as the fast dynamics of the system and the long term variation described by changes in the central value of Q as the slow dynamics. In this sense Figure 1 shows a system in (stochastic) equilibrium on the upper branch of the S curve near $Q = 1$ and the δ parameter slowly varies over time in the negative direction until the equilibrium branch it is on collapses, at which point the system rapidly moves to the only equilibrium point available near $Q = -1$. The hysteresis of the type shown here has been studied in physics [15], economics [7], abrupt climate change [16], the global dynamics of share market crashes [17] and even in foraging models of ant collonies [18].

3 Local and global optimisation in a simple economy

The quantal response equilibrium (QRE) is an economic equilibrium concept developed by McKelvey and Palfrey [19,20]. Recently it has been adopted as a model of "bounded rationality" for economics [21,22] and a recent alternative derivation of the QRE was reported in [7] where the maximum entropy [6] (MaxEnt) technique was used. We begin with this recent derivation using a dual Lagrangian approach arriving directly at the QRE and then show the extension of this previous work to the global optimisation case using an alternative single Lagrangian.

We consider a two person non-cooperative game theory scenario in which the utility matrices for two economic agents a and b have incentives for making choices $i \in \{1, 2\}$ (a) and $j \in \{1, 2\}$ (b) whose value depends on the choices of the other agent. This is described using the two utility matrices given by:

$$u^a_{i,j} = \begin{bmatrix} u^a_{1,1} & u^a_{1,2} \\ u^a_{2,1} & u^a_{2,2} \end{bmatrix}, \quad u^b_{i,j} = \begin{bmatrix} u^b_{1,1} & u^b_{1,2} \\ u^b_{2,1} & u^b_{2,2} \end{bmatrix}$$

and the expected utility to each economic agent in each group is described in terms of the joint probability distribution $p(a = i)q(b = j) = p_i q_j$:

$$E(u^a) = \sum_{i,j} p_i q_j u^a_{i,j}, \quad E(u^b) = \sum_{i,j} p_i q_j u^b_{i,j}.$$

The conventional Nash equilibrium [23] is found by each agent maximising their expected utility by varying their distribution over strategies:

$$p^*_i = \mathrm{argmax}_{p_i} \sum_{i,j} p_i q_j u^a_{i,j} \,\forall\, i$$

and

$$q^*_j = \mathrm{argmax}_{q_j} \sum_{i,j} p_i q_j u^b_{i,j} \forall j.$$

The QRE is different in that it can be viewed as the solution to a constrained optimisation of the entropy [6,7] of each player's distribution:

$$p_i^* = \max_{p_i} S(p_i) = \max_{p_i}\left(- \sum_i p_i \ln(p_i)\right) \quad (1)$$

subject to the constraints:

$$p_i \geq 0 \;\forall\, i,\; \sum_i p_i \;=\; 1,\; \sum_{i,j} p_i q_j u^a_{i,j} \;=\; E\left(u^a\right). \quad (2)$$

Such a constrained optimisation problem can be solved by forming the Lagrangian and then finding the stationary solutions $\nabla_{\{p_i\}}\mathcal{L}(p_i) = 0$, see [24] Chapter 12 for details and a proof that these stationary solutions are (local) maximums:

$$\mathcal{L}(q_i) = S(p_i) + \beta_a \sum_{i,j} p_i q_j u^a_{i,j} + \beta_0 \sum_i p_i, \quad (3)$$

$$\frac{\partial \mathcal{L}(p_i)}{\partial p_i} = -\ln(p_i) + \beta_a \sum_j q_j u^a_{i,j} + \beta_0 - 1 = 0, \quad (4)$$

$$p_i = \mathcal{Z}_a^{-1} \exp\left(\beta_a \sum_j q_j u^a_{i,j}\right), \quad (5)$$

where the $\mathcal{Z}_a^{-1} = \exp(\beta_0 - 1)$ term enforces the normalisation constraints. This Lagrangian technique balances the rate of change due to the multiple constraints placed on the system. Note that if there were no constraints and we were to simply maximise the entropy a uniform distribution would result. Adding a constraint such as the expected utility requires the change in a players utility to be balanced against changes in the system's entropy as the probabilities vary. The β terms then act to either emphasise or de-emphasise the "cost" associated with these constraints. Because the β_0 term enforces the normalisation constraint it is entirely decided by the other constraints and the requirement that we want a probability distribution. It also follows that:

$$q_i = \mathcal{Z}_b^{-1} \exp\left(\beta_b \sum_i p_i u^b_{i,j}\right). \quad (6)$$

Note that equations (5) and (6) are identical with the QRE of MacKelvey and Palfrey [19]. The exponents are called the conditional expected utilities, for example $E(u^a|i) = \sum_j q_j u^a_{i,j}$ is the value player a receives if they play strategy i given the distribution of their opponent's strategy.

For the global optimisation we now assume that both players are interested in maximising their own utility plus the utility of the other player. It will be shown that the net effect is that both players have the same utility function, a situation common in many games [10,25,26]. The constrained optimisation is to maximise the entropy subject to a total expected utility:

$$\max_{p_i,q_j} S(p_i q_j) = \max_{p_i,q_j}\left(- \sum_{i,j} p_i q_j \ln(p_i q_j)\right) \quad (7)$$

subject to the constraints:

$$p_i \geq 0, \quad q_j \;\geq\; 0 \quad \forall\, i,j \quad (8)$$

$$\sum_i p_i = 1,\; \sum_j q_j \;=\; 1,\; \sum_{i,j} p_i q_j \;=\; 1, \quad (9)$$

$$\sum_i p_i q_j = q_j \;\forall\, j,\; \sum_j p_i q_j = p_i \;\forall\, i \quad (10)$$

$$E\left(u^a\right) + E\left(u^b\right) = \sum_{i,j} p_i q_j \left(u^a_{i,j} + u^b_{i,j}\right) = E\left(u^{tot}\right). \quad (11)$$

The constraints in equation (9) implicitly requires that the p_i and q_j be chosen independently of each other. The Lagrangian of the joint probability distribution is:

$$\begin{aligned}\mathcal{L}(p_i q_j) = {} & S(p_i q_j) + \beta \sum_{i,j} p_i q_j \left(u^a_{i,j} + u^b_{i,j}\right) \\ & + \sum_j \left(\delta^a_j \left(\sum_i p_i q_j - q_j\right)\right) \\ & + \sum_i \left(\delta^b_i \left(\sum_j p_i q_j - p_i\right)\right) \\ & + \beta_0 \sum_{i,j} p_i q_j + \beta_0^a \sum_i p_i + \beta_0^b \sum_j q_j. \end{aligned} \quad (12)$$

Note that there are two unusual terms here: $\sum_j \left(\delta^a_j \left(\sum_i p_i q_j - q_j\right)\right)$ and $\sum_i \left(\delta^b_i \left(\sum_j p_i q_j - p_i\right)\right)$. For all of the other constraints in the Lagrangian one side of the constraint is a constant and is then lost when we differentiate in the next step, but this is not the case for the constraints in equation (10). This might complicate the solution we arrive at but it will be shown below that by fixing the δ^a_j and δ^b_i values we can cancel these terms with an additional entropic term that arises here but is not present in the previous dual Lagrangian approach.

The β_0 term enforces the normalisation constraints of the joint distribution over choices (Eq. (9)), the δ^a_j and δ^b_i parameters enforce the marginalisation constraints (Eq. (10)) and the β term enforces the constraint on the total utility $E(u^{tot})$ (Eq. (11)). The stationary points are found by setting $\nabla_{\{p_i,q_i\}}\mathcal{L}(p_i q_j) = 0$ and solving for p_i and q_j:

$$\begin{aligned}\frac{\partial \mathcal{L}(p_i q_j)}{\partial p_i} = {} & S(q_j) - \ln(p_i) + \beta \sum_j q_j (u^a_{i,j} + u^b_{i,j}) \\ & + \sum_j \delta^a_j q_j + \beta_0 + \beta_0^a - 1 = 0 \end{aligned} \quad (13)$$

$$\begin{aligned}\frac{\partial \mathcal{L}(p_i q_j)}{\partial q_j} = {} & S(p_i) - \ln(q_j) + \beta \sum_i p_i (u^a_{i,j} + u^b_{i,j}) \\ & + \sum_i \delta^b_i p_i + \beta_0 + \beta_0^b - 1 = 0. \end{aligned} \quad (14)$$

Note that by setting $\delta^a_j = \ln q_j \;\forall\, j$ and $\delta^b_i = \ln p_i \;\forall\, i$ then $S(q_j) + \sum_i \delta^a_j q_j = 0$, $S(p_i) + \sum_j \delta^b_i p_i = 0$ and the fixed points are:

$$p_i = \mathcal{Z}_a^{-1} \exp\left(\beta \sum_j q_j \left(u_{i,j}^a + u_{i,j}^b\right)\right) \tag{15}$$

$$q_j = \mathcal{Z}_b^{-1} \exp\left(\beta \sum_i p_i \left(u_{i,j}^a + u_{i,j}^b\right)\right) \tag{16}$$

and $\mathcal{Z}_a^{-1} = \exp(\beta_0 + \beta_0^a - 1)$ and $\mathcal{Z}_b^{-1} = \exp(\beta_0 + \beta_0^b - 1)$ enforce the normalisation constraints in equation (9). There are two significant points of differentiation between equations (5) and (6) compared to equations (15) and (16). The first is that there is only one β term used for both p_i and q_j in equations (15) and (16). The second is that the utilities that appear in the exponents in equations (15) and (16) are the sum of the utilities rather than the individual utilities. Taken together, these two points mean that there is only one overall utility matrix that both players use: $u_{i,j} = u_{i,j}^a + u_{i,j}^b$ and a single β parameter that is equivalent to $\beta = \beta_a = \beta_b$ in equations (5) and (6). In what follows we refer to the fixed points found using the local Lagrangian QRE_L and the fixed points of the global Lagrangian QRE_G.

4 Fixed point surfaces, bifurcations and adaptive flexibility

In order to simplify the discussion we reduce the QRE and the Global QRE by substituting $Q_a = 1 - 2q_i$, $Q_a \in [-1, 1]$ and $Q_b = 1 - 2p_j$, $Q_b \in [-1, 1]$. In the following examples we discuss the results exclusively in terms of the behavioural variable Q_b. For the simple examples shown here this is not a restriction as the same discussion would hold for Q_a, however in more complex games this is not necessarily the case. The system's equilibrium points are then determined by the payoff matrices and the four terms: Q_a, Q_b, β_a and β_b.

The first game we consider is:

$$u_{i,j}^a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad u_{i,j}^b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The fixed point surface for Q_b is shown in the upper plot of Figure 2. The plane that passes through the plot at $\beta_a = \beta_b$ intersects the fixed point QRE surface as shown in lower plot of Figure 2. The utility matrix for the Global QRE is the utility of a (or b) rescaled by a factor of 2, this shifts where the bifurcation point occurs in the lower plot of Figure 2 (from $\beta = 2$ to $\beta = 1$) but not the qualitative features. So the lower plot of Figure 2 showing the intersection of the QRE with the plane $\beta_a = \beta_b$ has the same qualitative features as the Global QRE for this game. In this sense QRE_G is qualitatively a sub-space of QRE_L. Note that if the system is any equilibrium state on a branch above the pitchfork bifurcation, if either β_a, β_b (top plot) or β (bottom plot) continuously decrease there is no way in which the system can avoid going through the bifurcation point.

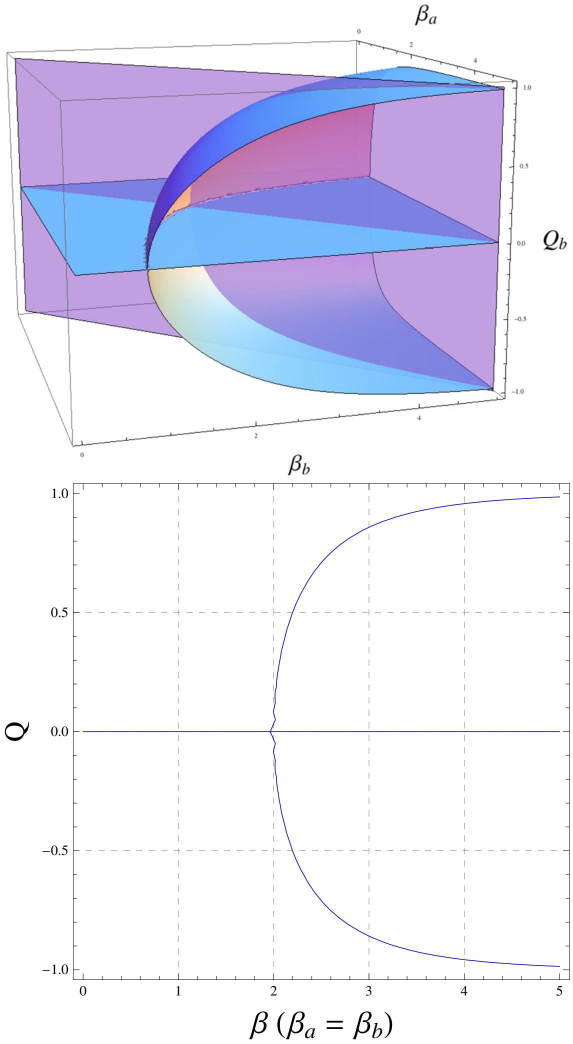

Fig. 2. Top: the equilibrium surface for the first example showing a generalisation of the pitchfork bifurcation. Bottom: the intersection of the plane that runs through the plot at 45°, i.e. $\beta_a = \beta_b$, in the plot above with the QRE_G surface showing the subspace of fixed points for (a scaled form of) QRE_G showing a conventional pitchfork bifurcation.

The second game is a perturbed version of the first:

$$u_{i,j}^a = \begin{bmatrix} 1 & 0 \\ 0 & 1.1 \end{bmatrix}, \quad u_{i,j}^b = \begin{bmatrix} 1 & 0 \\ 0 & 1.1 \end{bmatrix}.$$

In this case QRE_G is again qualitatively a sub-space of QRE_L but in this case the bifurcation involves a discontinuous jump in the behavioural variable Q_b as seen in Figure 3. In this case a large portion of the QRE_L surface is isolated from the continuous upper surface. Some earlier results [27] suggest that this might be a localised effect in the sense that as either $\beta_a \to \infty$ or $\beta_b \to \infty$ the two surfaces meet.

The third game is a qualitatively different perturbation:

$$u_{i,j}^a = \begin{bmatrix} 1 & 0 \\ 0 & 0.95 \end{bmatrix}, \quad u_{i,j}^b = \begin{bmatrix} 1 & 0 \\ 0 & 1.05 \end{bmatrix}.$$

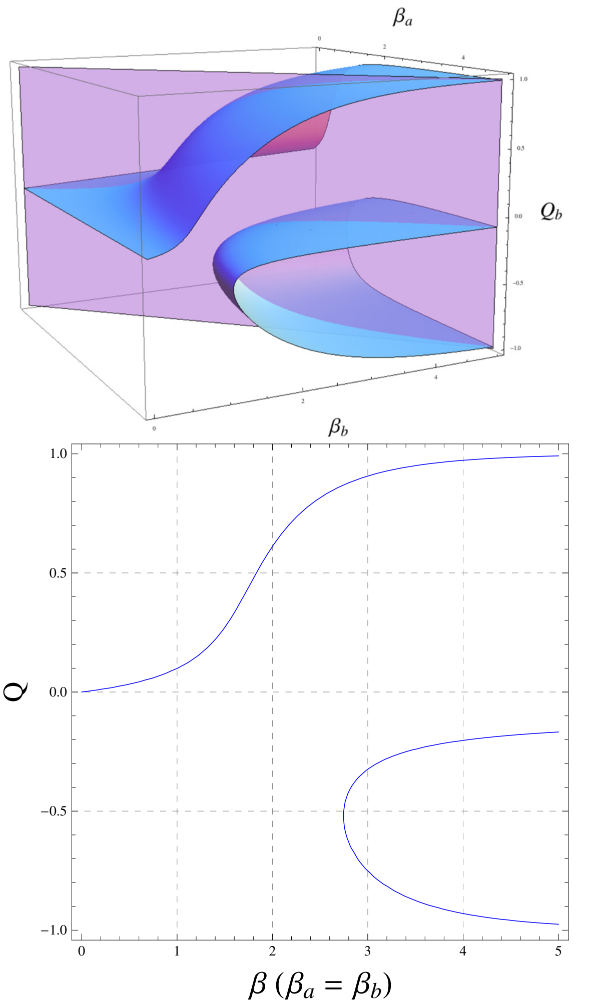

Fig. 3. Top: the QRE_L surface for the second game showing the local separation of one equilibrium surface from another. Bottom: the 45^o β plane that intersects the QRE_L surface in the plot above.

In this example the QRE_G is still a sub-space of the rescaled QRE_L with a utility matrix for both players in this example for QRE_G is:

$$u_{i,j} = u^a_{i,j} + u^b_{i,j} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

i.e. the same (rescaled) matrix for QRE_G as in the first example that resulted in a pitch-fork bifurcation. Now however QRE_L has a far more complex shape as shown in Figure 4 where the $\beta_a = \beta_b$ plane has been omitted for clarity, however it does intersect QRE_L in a pitchfork bifurcation just as in the lower plot of Figure 2.

Now there is no separation of the two surfaces, instead there is a *twist* in the surface topology that exchanges the connective paths between the upper and lower branches. Because of the symmetry of the utility matrices the twist occurs precisely where the rescaled QRE_G sub-space intersects QRE_L. But in the case of QRE_L there is the potential for β_a and β_b to vary at different rates. As $\beta = \beta_a = \beta_b$ is a requirement of QRE_G there is no flexibility in the

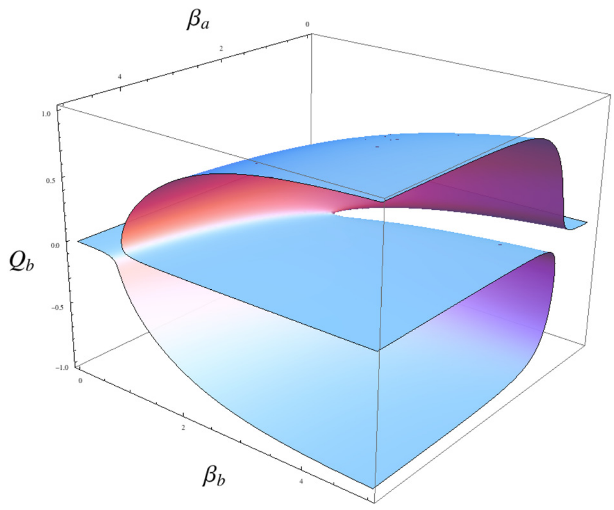

Fig. 4. The equilibrium surface for QRE_L in example 3.

relationship between β_a and β_b, however for QRE_L this is not the case and we can posit two different functional relationships between these two parameters: $\beta_a = f(\beta_b)$ and $\beta_a = g(\beta_b)$. While we describe functional relationships $f(\beta_a)$ and $g(\beta_b)$ we note that these are very general considerations. The purpose is only to show that if both β_a and β_b are under the explicit control of an external agent, then perturbing one term relative to the other enables tipping points to be avoided, see reference [7] for an example where these β terms are tax rates set by an external government agency. These alternative surfaces through QRE_L are shown in Figure 5 and the subsequent subspaces are shown in Figure 6. The resultant fixed point curves for these alternative joint variations in β_a and β_b clearly show that should a single parameter, either β_a or β_b, be slowly decreasing a tipping point can be avoided irrespective of which branch of the equilibrium surface the system is on (either the top branch or the lower branch), there is a parameter variation that leads to a perfectly smooth transition of the system without passing through a tipping point.

5 Discussion

To illustrate these ideas we consider an idealised example. The automotive industry might produce a combination of petrol and electric powered cars and the energy sector might build a combination of outlets that sell either electricity or petrol to car drivers. These are "independent strategies" each industry arrives at based on the underlying incentives, but the incentives for producing a particular type of car changes depending on the availability of the different types of stations: if more electricity stations are built it makes it more practical to have an electric car rather than a petrol car. These microeconomic dynamics that are based on the underlying incentives are usually described using stochastic differential equations (SDEs) and there are many game theoretical results arriving at an exponential form of the probability

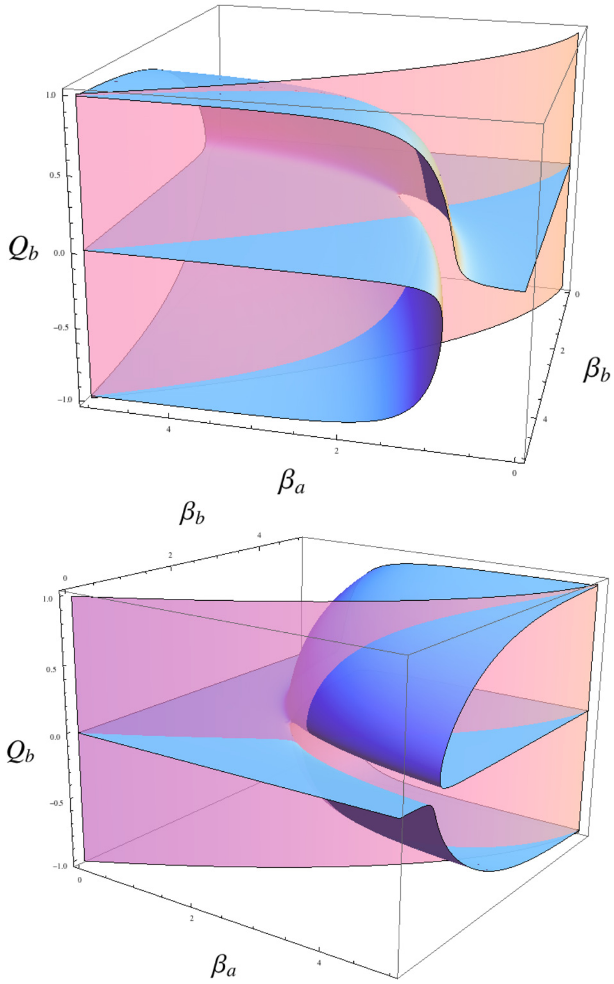

Fig. 5. Two alternative subspaces of QRE_L can be found by a variation in the relationship between β_a and β_b, top: $\beta_a = f(\beta_b)$ and bottom: $\beta_a = g(\beta_b)$ for the same QRE_L of example 3, see Figure 6 for the equilibrium points in these two sub-spaces. Note that the curvature of the intersecting surface is in the opposite direction when comparing the top and bottom plots, hence different functional relationships $f(\beta_b)$ and $g(\beta_b)$. A suitable choice of such a variation allows for the navigation around a tipping point irrespective of where the system happens to be initially, either on the top branch or the bottom branch.

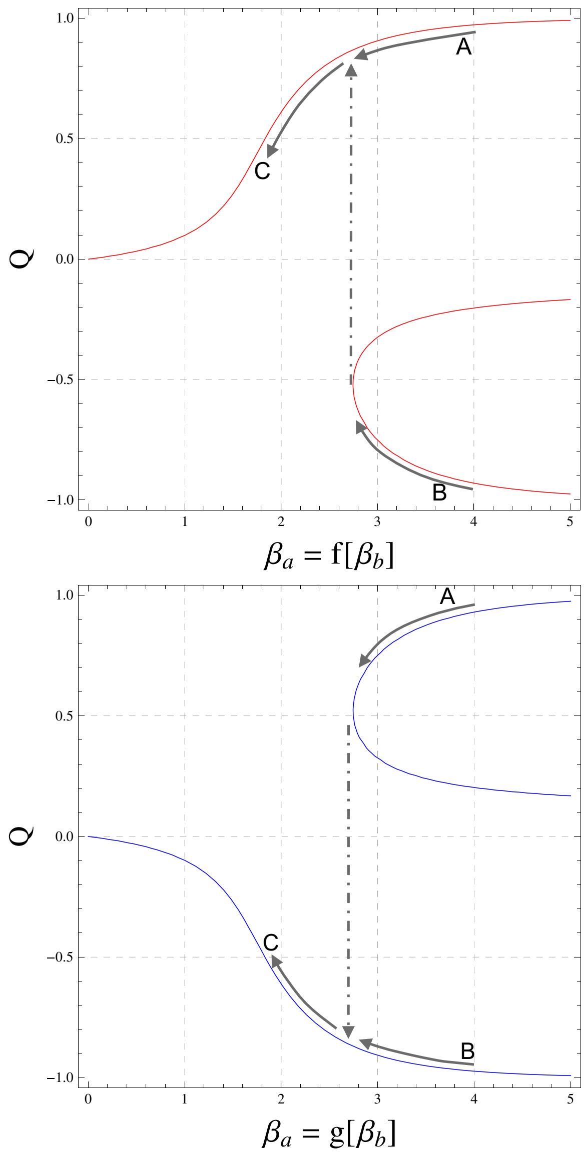

Fig. 6. The equilibrium points of the sub-spaces that intersect QRE_L in the top and bottom plots of Figure 5. Note that depending on where the system starts, either on the top branch (A) or the bottom branch (B), a perturbation to the relationship between β_a and β_b through choosing either $f(\beta_b)$ or $g(\beta_b)$ results in a smooth trajectory from the starting point, either A or B, to position C without passing through a tipping point.

distributions [5,8,9,28,29]. In this sense, changing the incentives changes the evolutionary dynamics of the SDEs, and the different incentives are then responsible for the different structures of the tipping points.

Such considerations lead to a key result relating SDEs to the MaxEnt Lagrangian: if there is an SDE that models a system's micro-dynamics with linear drift terms (linear expectations across incentives in game theory) there is a corresponding MaxEnt formulation, both of which have the same stationary solutions [30]. While the results described in this article are not explicitly based on a model of micro-dynamics, there exists a system of micro-dynamics for which the probability distributions derived here are an equilibrium solution. So from this point of view, even though we have maximised the entropy of a system subject to some constraints, the result is that of an equilibrium distribution of a related micro-dynamical system. Whether or not this other system reflects a useful economic dynamic is the subject of future work.

Tipping points are one of the most important diagnostic tools we have available to study complex environments such as financial markets [7,31] and climate systems [32–34] and both of these areas use models of varying levels of detail. Simplified models of coupled non-linear systems such as the two presented here provide an important step in improving our understanding of the nature and potential for navigating around such catastrophes. Considerable research effort is being applied to understanding the direction a system is heading in and its proximity to a tipping point, with a particular focus on climate change [35–38]. In principle at least, if an external "system manager" knew where a system was within its phase space and their models were based on adequate empirical knowledge of that system, then it is possible to have some degree of security in their ability to manage the task of avoiding a tipping point.

There are three conclusions that may be drawn at this point. The first is that tipping points such as these are sensitively dependent on all of the system parameters. As the system parameters change bifurcations occur and the potential exists for the system to pass through a tipping point. It is important to note that for any normal form game there is a single unique fixed point when all Lagrange parameters are set to zero and the number of fixed points for non-zero Lagrange parameters is bounded by the number of Nash equilibria there are in the original game [19].

The second is that relative perturbations to system parameters are sufficient to avoid catastrophes. This is in sharp contrast to an idea implicit in much work on tipping points that a system's trajectory might need to be stopped or even reversed in order to avoid a catastrophe. What we have shown here is that a suitable choice of perturbation to the *relative rate of change* of system parameters is sufficient to avoid passing through a region in which fixed points collapse and significantly new and distant fixed points need to be reached.

A final remark is in order on the nature of the MaxEnt procedure. Note that for a given expected utility $E(u^a)$ for one of the agents, the resultant QRE_L for that agent is the *flattest possible distribution* consistent with $E(u^a)$, as measured by the entropy. A more general case is one in which instead of a single agent there is a market sector of the economy that can produce many different products where each type represents a strategy, the quantity produced represents a distribution over these strategy types and the expected utility is some measure of economic output such as the industry's contribution to a nation's Gross Domestic Product [39]. In this case the MaxEnt distribution provides for the greatest variety of products consistent with a given economic output. In this sense it maximises the Shannon Diversity Index (SDI) [40] subject to the expected economic output of the market sector. The SDI is usually used in measuring species diversity in ecosystems, suggesting an intriguing parallel between tipping points, diversity and resilience in economics just as there is in ecology research and its related issues with climate change tipping points [41].

Moreover, evidence of bifurcations have been found in financial markets [42] and macro-economics [43]. Recent evidence from econo-physics has also shown the signatures of phase-transition like behaviour in both the macroscopic [44] and microscopic [45] interactions in financial markets. Such macro-economic systems, controlled through variations in tax regimes, has already been proposed [7] and provides a pragmatic situation in which control parameters can be manipulated. What has not been previously addressed is the issue of tipping points and their management. Early warning signatures of tipping points are detectable before a catastrophic crash [36] and so the possibility of navigating coupled non-linear systems such as an economy around tipping points is one of the key goals of such system level analysis.

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